Int. J. Solids Structures Vol. 27, No. 7, pp. 825–832, 1991
Printed in Great Britain.

0020-7683/91 \$3.00 + .00
Pergamon Press plc

DETERMINATION OF PERIODIC OSCILLATIONS IN NONLINEAR AUTONOMOUS DISCRETE–CONTINUOUS SYSTEMS WITH DELAY

J. AWREJCEWICZ†
Technical University of Łódź, Institute of Applied Mechanics, B. Stefanowskiego 1/15, 90-924 Łódź, Poland

(*Received* 17 *January* 1989; *in revised form* 8 *December* 1989)

Abstract—This paper presents an analytical method of determining the periodic solutions in mechanical discrete–continous systems governed by a system of nonlinear ordinary and partial differential equations with delay. Contrary to the classical perturbation method, the solutions are sought as power series in relation to two small independent parameters. One of the parameters is related to nonlinearity, the other, to delay.

1. INTRODUCTION

The classical perturbation method, which consists of seeking periodic solutions in the form of power series in relation to arbitrarily chosen small parameters, is broadly dealt with in Malkin (1956), Iakubovich and Starzhinskii (1972), Giacaglica (1972), Nayfeh and Mook (1979) and Nayfeh (1981). It allows for the determination of periodic solutions for systems described by nonlinear ordinary differential equations with constant or periodically varying coefficients. The method is based on bringing various nonlinearities of parametric excitation to one small parameter. However, in real mechanical systems the parameters describing nonlinearities or parametric excitations are independent, and the results of formally bringing them to one parameter are not always satisfactory.

This paper deals with mechanical discrete–continuous systems with delay as well as with the occurrence of various kinds of physical or geometric nonlinearities. An example of such a system is a furnace, where the temperature is controlled by a thermoregulator. The furnace is a nonlinear continuous system while the thermoregulator is usually an inertial element with delay (a discrete system). Another example is the nonlinear vibration of beams joined to mechanical discrete systems. In this case, delay is characterized by inertial friction in the materials. These types of nonlinear discrete–continuous systems are governed by nonlinear ordinary and partial differential equations with delay. These equations are the subject of this paper, which is the continuation of the author's previous studies covering nonlinear discrete systems based on an analytical approach (Awrejcewicz, 1986, 1988).

2. METHOD

Consider the system of equations of the form

$$\frac{\partial^2(t_1,x)}{\partial t_1^2} = c^2\frac{\partial^2 u(t_1,x)}{\partial x^2} + f\left(\varepsilon, x, u(t_1,x), \frac{\partial u(t_1,x)}{\partial t_1}, \frac{\partial u(t_1,x)}{\partial x}, y(t_1-\tau)\right)$$

$$L_1[y(t),\tau_r] = \varphi[\varepsilon, y(t_1), u(t_1-\tau,\xi)] \qquad (1)$$

with homogeneous boundary conditions

† Presently with: The Technical University of Braunschweig, Institute of Technical Mechanics, Spielmannstraße 11, D-3300 Braunschweig, F.R.G.

$$u(t_1, 0) = u(t_1, 1) = 0, \tag{2}$$

where:

f is a certain nonlinear function assuming zero for $x = 0$ and $x = l$;

$L_1(y, \tau_r)$ is a linear differential operator with delay of the form

$$L_1[y, \tau_r] = \sum_{p=0}^{P} \sum_{r=0}^{R} a_{pr} y^{(p)}(t_1 - \tau_r), \quad \tau_0 = 0, \quad \tau_r > 0;$$

φ is a nonlinear differential operator with delay, which is because $y(t)$ is of order lower than P, and considering $u(t - \tau, \xi)$ of order not higher than two, with $\xi \in [0, 1]$.

Next we assume that the nonlinear operators f and φ are continuous in x with continuous first derivatives, considering other arguments in a certain sufficiently large range of their variations. Moreover, we assume that the delays occurring in the system are small. Thus we have

$$y(t_1 - \tau) = y(t_1) - \tau \frac{dy(t_1)}{dt_1} + \frac{1}{2}\tau^2 \frac{d^2y(t_1)}{dt_1^2} \dots,$$

$$u(t_1 - \tau, \xi) = u(t_1, \xi) - \tau \frac{\partial u(t_1, \xi)}{\partial t_1} + \frac{1}{2}\tau^2 \frac{\partial^2 u(t_1, \xi)}{\partial t_1^2} \dots \tag{3}$$

Further calculations will be limited only to the first three terms of the series (3). Substituting (3) in (1), we obtain

$$\frac{\partial^2 u(t_1, x)}{\partial t_1^2} = c^2 \frac{\partial^2 u(t_1, x)}{\partial x^2} + f_1\left(\varepsilon, x, u(t_1, x), \frac{\partial u(t_1, x)}{\partial t_1}, \frac{\partial u(t_1, x)}{\partial x}, \tau, y, \frac{dy}{dt_1}, \frac{d^2y}{dt_1^2}\right),$$

$$L_1[y(t_1), \tau_r] = \varphi_1\left(\varepsilon, y(t_1), \tau, u(t_1, \xi), \frac{\partial u(t_1, \xi)}{\partial t_1}, \frac{\partial^2 u(t_1, \xi)}{\partial t_1^2}\right), \tag{4}$$

where the functions f_1 and φ_1 are obtained respectively from f and φ, considering (3).

Let the nonlinear functions f_1 and φ_1 assume zero when $\varepsilon = \tau = 0$, which means that the nonlinear system of differential equations (4) is then reduced to a linear system.

Then, the problem becomes the analysis of the system of equations (4) with two independent small parameters τ and ε.

Let us further assume that the characteristic equation adequate for the linear part of the second equation of the system is of the form

$$\Theta(\rho) = \sum_{p=0}^{P} \sum_{r=0}^{R} a_{pr} \rho^p e^{-\tau_r \rho}, \tag{5}$$

and that its eigenvalues are different from zero and have purely imaginary values. This means that oscillations are not generated by the discrete system. The starting solution for the analytical approximate method, with $\varepsilon = 0$, $\tau = 0$ is of the form

$$U_{(*)}^0(t_1, x) = \sum_{n=1}^{\infty} \sin\frac{\pi x}{l} [a_{(*)n}^0 \cos(n\alpha_0 t_1) + b_{(*)n}^0 \sin(n\alpha_0 t_1)], \quad y_{(*)}^0(t_1) = 0 \tag{6}$$

where the operator $(*)$ denotes τ or ε, $a_{(*)n}^0$ and $b_{(*)n}^0$ are amplitudes, and $T_0 = 2\pi/\alpha_0 = 2l/c$ is the period of oscillation of the linear part of the system described by the first equation (4). For $\varepsilon \neq 0$ and $\tau \neq 0$ in a satisfactorily close neighbourhood of zero, we seek the periodic solution of the system (4) a little different from (6). Generally, the contribution of higher harmonics to the solution quickly decreases, and it is sufficient to consider only a few of the first harmonics in the calculations. The period sought is equal to

$$T = T_0[1+\eta(\varepsilon,\tau)] \tag{7}$$

and evidently depends on both of the perturbation parameters.

Let us introduce a new dimensionless time t according to the equation

$$t_1 = \frac{1+\eta(\varepsilon,\tau)}{\alpha_0}t, \tag{8}$$

which allows us to seek a periodic solution with period 2π.

Substituting (8) in (4), we obtain the equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \left(\frac{1}{\pi}\right)^2 \frac{\partial^2 u(t,x)}{\partial x^2} + F\left(\varepsilon, x, u(t,x), \frac{\partial u(t,x)}{\partial t}, \frac{\partial u(t,x)}{\partial x}, \tau, y, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right)$$

$$L[y(t),\tau_r] = \phi\left(\varepsilon, y(t), t, u(t,\xi), \frac{\partial u(t,x)}{\partial t}, \frac{\partial^2 u(t,x)}{\partial t^2}\right), \tag{9}$$

where:

$$F = \frac{(1+\eta^2)l^2}{\pi^2 c^2} f_1,$$

$$L = \sum_{p=0}^{P}\sum_{r=0}^{R} a_{pr}\left\{\left[\frac{l(1+\eta)}{\pi c}\right]^2 y^{(p)}(t) - \tau_r \frac{l(1+\eta)}{\pi c} y^{(p+1)}(t) + \frac{1}{2}\tau_r^2 y^{(p+2)}(t)\right\},$$

$$\phi = \left[\frac{l(1+\eta)}{\pi c}\right]^2 \varphi.$$

The nonlinear functions ϕ and F as well as the solutions sought, y, u and η, are presented in the form of power series

$$\phi = \phi_0 + \varepsilon\phi_\varepsilon + \varepsilon^2\phi_{\varepsilon\varepsilon} + \cdots + \tau\phi_\tau + \tau^2\phi_{\tau\tau} + \cdots + \tau\varepsilon\phi_{\tau\varepsilon} + \cdots,$$

$$F = F_0 + \varepsilon F_\varepsilon + \varepsilon^2 F_{\varepsilon\varepsilon} + \cdots + \tau F_\tau + \tau^2 F_{\tau\tau} + \cdots + \tau\varepsilon F_{\tau\varepsilon} + \cdots,$$

$$y = y_0 + \varepsilon y_\varepsilon + \varepsilon^2 y_{\varepsilon\varepsilon} + \cdots + \tau y_\tau + \tau^2 y_{\tau\tau} + \cdots + \tau\varepsilon y_{\tau\varepsilon} + \cdots,$$

$$u = u_0 + \varepsilon u_\varepsilon + \varepsilon^2 u_{\varepsilon\varepsilon} + \cdots + \tau u_\tau + \tau^2 u_{\tau\tau} + \cdots + \tau\varepsilon u_{\tau\varepsilon} + \cdots,$$

$$\eta = \eta_0 + \varepsilon\eta_\varepsilon + \varepsilon^2\eta_{\varepsilon\varepsilon} + \cdots + \tau\eta_\tau + \tau^2\eta_{\tau\tau} + \cdots + \tau\varepsilon\eta_{\tau\varepsilon} + \cdots. \tag{10}$$

Having substituted (10) in (9), and after having equated the expressions representing the same powers of the small parameters τ and ε as well as the same powers of their products $\tau^m\varepsilon^l (m, l = 1, 2, \ldots)$, the recurrent systems of linear equations are obtained. While solving the subsequent equations of the system, we use the balance of harmonics method. Let us assume that we determined the first system of recurrent equations standing next to (*), where the operator (*) means τ or ε. Having substituted the solutions (6) for the nonlinear functions $F_{(*)}$ and $\phi_{(*)}$ (this time for the equation we assume $t_1 = t$ and $\alpha_0 = 1$) and having developed these functions into a Fourier series, we obtain

$$F_{(*)}(t,x) = \sum_{p=0}^{\infty}\sum_{k=0}^{\infty} \sin\frac{n\pi x}{l}[A_{nk}^{(*)}\cos(kt) + B_{nk}^{(*)}\sin(kt)],$$

$$\phi_{(*)}(t) = \sum_{k=0}^{\infty}[C_k^{(*)}\cos(kt) + D_k^{(*)}\sin(kt)], \tag{11}$$

where:

$$A_{nk}^{(*)} = \frac{2}{\pi l}\int_0^l \int_0^{2\pi} F_{(*)}(t,x)\sin\frac{n\pi x}{l}\cos(kt)\,dt\,dx,$$

$$B_{nk}^{(*)} = \frac{2}{\pi l}\int_0^l \int_0^{2\pi} F_{(*)}(t,x)\sin\frac{n\pi x}{l}\sin(kt)\,dt\,dx,$$

$$C_k^{(*)} = \frac{1}{\pi}\int_0^{2\pi} \phi_{(*)}(t)\cos(kt)\,dt,$$

$$D_k^{(*)} = \frac{1}{\pi}\int_0^{2\pi} \phi_{(*)}(t)\sin(kt)\,dt. \tag{12}$$

We seek the solutions of the system of equations formed by comparison of expression next to (*) in the form of

$$U_{(*)}(t,x) = \sum_{n=1}^{N}\sum_{k=0}^{K} \sin\frac{n\pi x}{l}[a_{(*)nk}\cos(kt)+b_{(*)nk}\sin(kt)],$$

$$y_{(*)}(t) = \sum_{k=0}^{K}[c_{(*)k}\cos kt + d_{(*)k}\sin(kt)]. \tag{13}$$

The solution of the first equation of system (9) is explicitly determined only when

$$P_{(*)n}(a_{(*)s}^0, b_{(*)s}^0, \eta_c) = \int_0^l \int_0^{2\pi} F_{(*)}(t,x)\sin\frac{n\pi x}{l}\cos(nt)\,dt\,dx = 0,$$

$$Q_{(*)n}(a_{(*)s}^0, b_{(*)s}^0, \eta_c) = \int_0^l \int_0^{2\pi} F_{(*)}(t,x)\sin\frac{n\pi x}{l}\sin(nt)\,dt\,dx = 0. \tag{14}$$

Conditions (14) allow for neglecting the resonance terms which exponentially grow with time (Malkin, 1956). Thus we obtain $2N$ of the equations whereas the unknowns $a_{(*)s}^0$, $b_{(*)s}^0$ and η_c are $2N+1$. In this case, however, dealing with an autonomous system, we may assume that $b_{(*)N} = 0$. Equations (14) have explicit solutions when

$$\frac{\partial(P_{(*)1}, P_{(*)2}, \ldots, P_{(*)N}, Q_{(*)1}, Q_{(*)2}, \ldots, Q_{(*)N})}{\partial(a_{(*)1}^0, a_{(*)2}^0, \ldots, a_{(*)N}^0, b_{(*)1}^0, b_{(*)2}^0, \ldots, b_{(*)N-1}^0, \eta_c)} \neq 0. \tag{15}$$

3. EXAMPLE

Consider the discrete-continuous system described by the equations

$$\frac{\partial^2 u}{\partial t_1^2} = \left(\frac{30}{\pi}\right)^2 \frac{\partial^2 u(t_1,x)}{\partial x^2} + \varepsilon[0.003 - u^2(t_1,x)]\frac{\partial u(t_1,x)}{\partial t_1} + \varepsilon\delta(x-\bar{x})y(t_1)$$

$$-\varepsilon\tau\delta(x-\bar{x})\frac{dy}{dt_1} + \tau\left(\frac{\partial u(t_1,x)}{\partial t_1} - \frac{\partial^2 u(t_1,x)}{\partial x^2}\right) - \tau\left(\frac{\partial u(t_1,x)}{\partial t_1}\right)^3,$$

$$\frac{d^2 y}{dt_1^2} + 10\varepsilon\frac{dy}{dt_1} + 400y(t_1) = 10\varepsilon u(t_1,\bar{x}), \quad u(t_1,0) = u(t_1,1) = 0 \tag{16}$$

where, for the sake of simplification of the calculations, the delay τ and the small parameter ε are in the evident form in eqn (1) and $\bar{x} \in [0, 1]$ is the association point of the discrete system with the continuous one. In the discrete system described by the second equation of the system (16) accompanied by the lack of interaction on the side of the continuous system and as a result of damping in the system, oscillations cannot occur. The oscillations are excited in the continuous system because of damping of the Van der Pol type described by the second term on the right-hand side of the equality sign. For $\tau = \varepsilon = 0$ the period of this solution is equal to $T_0 = \pi/15$. We seek the periodic solution of system (16) with period T, inconsiderably different from the period T_0. According to the considerations, let us first make use of the independent variable

$$t_1 = \frac{1+\eta(\varepsilon,\tau)}{30} t. \tag{17}$$

Having substituted (17) in (16), we obtain

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{1}{\pi^2}(1+\eta)^2 \frac{\partial^2 u(t,x)}{\partial x^2} + \varepsilon \frac{(1+\eta)}{30}[0.003 - u^2(t,x)]\frac{\partial u(t,x)}{\partial t} + \varepsilon \frac{(1+\eta)^2}{900}\delta(x-\bar{x})y(t)$$

$$-\varepsilon\tau \frac{1+\eta}{30}\delta(x-\bar{x})\frac{dy}{dt} + \tau \frac{1+\eta}{30}\frac{\partial u(t,x)}{\partial t} - \tau \frac{(1+\eta)^2}{900}\frac{\partial^2 u(t,x)}{\partial x^2} - \tau \frac{30}{1+\eta}\left(\frac{\partial u(t,x)}{\partial t}\right)^3;$$

$$\frac{d^2 y}{dt^2} + \varepsilon \frac{1}{3}(1+\eta)\frac{dy}{dt} + \frac{4}{9}(1+\eta)^2 y(t) = \frac{\varepsilon}{90}(1+\eta)^2 u(t,\bar{x}). \tag{18}$$

The parameters τ and ε are treated as independent. Assuming one of them to be equal to zero, the problem is reduced to the classical perturbation method.

We assume the starting solution in the form of

$$u^{(0)} = u_\tau^{(0)} + u_\varepsilon^{(0)} = a_\varepsilon^{(0)} \sin \pi x \cos t + a_\tau^{(0)} \sin \pi x \cos t,$$
$$y^{(0)} = 0. \tag{19}$$

The amplitude sought, $a_\varepsilon^{(0)}$, will be determined from the first recurrence equation formed by the comparison of expressions next to the parameter ε, whereas the amplitude $a_\tau^{(0)}$ will be determined from the first recurrent equation formed by the comparison of expressions next to the parameter τ.

From the first equation of the system (18), having equated the expressions next to the parameters τ, we obtain

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2} + \frac{2\eta\tau}{\pi^2}\frac{\partial^2 u_\tau^{(0)}}{\partial x^2} + \frac{1}{30}\frac{\partial u_\tau^{(0)}}{\partial t} - \frac{1}{900}\frac{\partial^2 u_\tau^{(0)}}{\partial x^2} - 30\left(\frac{\partial u_\tau^{(0)}}{\partial t}\right)^3. \tag{20}$$

Having equated the resonance terms to zero, we obtain

$$\eta_\tau = 0.0055$$
$$a_\tau^{(0)} = 0.044. \tag{21}$$

The solution of eqn (20) is

$$u_\tau = \frac{3}{128}(a_\tau^{(0)})^3 \sin 3\pi x \sin t - \frac{3}{128}\sin \pi x \sin 3t + a_\tau \sin \pi x \cos t, \tag{22}$$

where the amplitude a_τ will be determined from the subsequent recurrent equation. This equation is of the form

$$\frac{\partial^2 u_{\tau\tau}}{\partial t^2} = \frac{1}{\pi^2}\frac{\partial^2 u_{\tau\tau}}{\partial x^2} + \frac{2}{\pi^2}\eta_\tau \frac{\partial^2 u_\tau}{\partial x^2} + \frac{2}{\pi^2}\eta_{\tau\tau}\frac{\partial^2 u_\tau^{(0)}}{\partial x^2} + \frac{1}{30}\eta_\tau \frac{\partial u_\tau^{(0)}}{\partial t} + \frac{1}{30}\frac{\partial u_\tau}{\partial t}$$
$$-\frac{1}{900}\frac{\partial^2 u_\tau}{\partial x^2} - \frac{2}{900}\eta_\tau \frac{\partial^2 u_\tau^{(0)}}{\partial x^2} - 90\frac{\partial (u^{(0)})^2 u_\tau}{\partial t} + 30\eta_\tau \frac{\partial (u_\tau^{(0)})^3}{\partial t}. \quad (23)$$

From eqn (23), having equated the resonance terms to zero, we obtain

$$-\frac{1}{30}\eta_\tau a_\tau^{(0)} - \frac{1}{30}a_\tau - \frac{9}{16}(a_\tau^{(0)})^2 a_\tau = 0,$$
$$-2\eta_\tau a_\tau - a_\tau^{(0)}\eta_{\tau\tau} + \frac{1}{900}a_\tau \pi^2 + \frac{2}{900}\eta_\tau a_\tau^{(0)}\pi^2 + \frac{135}{206}(a_\tau^{(0)})^5 - \frac{135}{8}\eta_\tau (a_\tau^{(0)})^3 = 0. \quad (24)$$

Solving the system of equations (24)

$$a_\tau = 0.00002,$$
$$\eta_{\tau\tau} = -0.00006. \quad (25)$$

From the second equation of system (18), we obtain

$$y_\tau = 0. \quad (26)$$

Let us now determine the perturbation equations formed due to the comparison of the expressions next to the parameter ε.

From the first equation of system (18), we obtain

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} = \frac{1}{\pi^2}\frac{\partial^2 u_\varepsilon}{\partial x^2} + \frac{1}{\pi^2}2\eta_\varepsilon \frac{\partial^2 u_\varepsilon^{(0)}}{\partial x^2} + \frac{1}{30}[0.003 - (u_\varepsilon^{(0)})^2]\frac{\partial u_\varepsilon^{(0)}}{\partial t} \quad (27)$$

and having equated the resonance terms to zero, we obtain a system of algebraic equations. Solving this, we have

$$a_\varepsilon^{(0)} = 0.12649$$
$$\eta_\varepsilon = 0. \quad (28)$$

The solution of (27) is

$$u_\varepsilon = a_\varepsilon \sin(\pi x)\cos t - 5.10^{-7}\sin 3\pi x \cos t. \quad (29)$$

From the second equation of system (18), we obtain

$$\frac{d^2 y}{dt^2} + \frac{4}{9}y_\varepsilon = \frac{1}{90}u_\varepsilon^{(0)}(t, x). \quad (30)$$

We seek the solution of eqn (30) in the form

$$y_\varepsilon = b_\varepsilon \cos t + c_\varepsilon \sin t. \quad (31)$$

Having substituted (31) in (30), we find

$$b_\varepsilon = -0.088\sin(\pi\bar{x}),$$
$$c_\varepsilon = 0. \tag{32}$$

From the second equation of the system (18), having equated the expressions next to ε^2, we obtain

$$\frac{d^2 y_{\varepsilon\varepsilon}}{dt^2} + \frac{4}{9}y_{\varepsilon\varepsilon} = \frac{1}{3}b_\varepsilon \sin t + \frac{1}{90}a_\varepsilon \sin(\pi\bar{x})\cos t. \tag{33}$$

We seek the solution of eqn (33) in the form

$$y_{\varepsilon\varepsilon} = b_{\varepsilon\varepsilon}\cos t + c_{\varepsilon\varepsilon}\sin t. \tag{34}$$

Having substituted (34) in (33), we calculate

$$b_{\varepsilon\varepsilon} = -\frac{\partial_\varepsilon}{50}\sin(\pi\bar{x}),$$
$$c_{\varepsilon\varepsilon} = -0.0048\sin\pi x. \tag{35}$$

From the first equation of system (18), having equated the expressions next to ε^2, we obtain

$$\frac{\partial^2 u_{\varepsilon\varepsilon}}{\partial t^2} = \frac{1}{\pi^2}\frac{\partial^2 u_{\varepsilon\varepsilon}}{\partial x^2} + \frac{2}{\pi^2}\eta_{\varepsilon\varepsilon}\frac{\partial^2 u_\varepsilon^{(0)}}{\partial x^2} + 0.0001\frac{\partial u_\varepsilon}{\partial t} - \frac{1}{30}(u_\varepsilon^{(0)})^2\frac{\partial u_\varepsilon}{\partial t} - \frac{1}{15}u_\varepsilon^{(0)}u_\varepsilon\frac{\partial u_\varepsilon^{(0)}}{\partial t} + y_\varepsilon\delta(x-\bar{x}). \tag{36}$$

From eqn (36) we finally calculate

$$a_\varepsilon = 0,$$
$$\eta_{\varepsilon\varepsilon} = -0.01\sin^2\pi\bar{x}. \tag{37}$$

Analogous calculations make it possible to determine the recurrent equations occurring with the combinations $\varepsilon^k\tau^l$ where $k \geqslant 1$ and $l \geqslant 1$.

4. CONCLUDING REMARKS

The method presented in the paper enables us to determine the periodic solutions in discrete-continuous systems described by nonlinear differential ordinary and partial equations with delay. Solutions as well as their period were sought in the form of power series in relation to two small independent parameters τ and ε. Assuming one of them to be equal to zero, the problem is reduced to the classical perturbation method.

Thanks to this method, the period of the equations was determined as a function of two parameters τ and ε. It enables certain optimization criteria to be realized; for example, ensuring a constant value for the period in a certain chosen range of changes of the parameters τ and ε, or the choice of parameters τ and ε so as to increase or decrease the frequency of vibration according to a previously-chosen criterion.

Acknowledgement—This work was supported by the Alexander von Humboldt Foundation.

REFERENCES

Awrejcewicz, J. (1986). System vibrations: rotor with self excited support. *Proc. Int. Conf. on Rotor Dynamics* (Tokyo), pp. 517–522.

Awrejcewicz, J. (1988). Determination of the limits of the unstable zones of the unstationary nonlinear mechanical systems. *Int. J. Nonl. Mech.* **23**(1), 87–91.

Giacaglia, G. E. (1972). *Perturbation Methods in Nonlinear Systems*. Springer, New York.

Iakubovich, V. A. and Starzhinskii, V. M. (1972). *Linear Differential Equations with Periodic Coefficients and their Applications* (in Russian). Nauka, Moscow.

Malkin, J. G. (1956). *Some Problems in the Theory of Nonlinear Oscillations* (in Russian). Gostekhizdat, Moscow.

Nayfeh, A. H. (1981). *Introduction to Perturbation Techniques*. John Wiley, New York.

Nayfeh, A. H. and Mook, P. T. (1979). *Nonlinear Oscillations*. John Wiley, New York.